SAYT1134 Towards Differential Geometry
Group 3 Tutorial 6

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1 Frenet frame

1. Let $\alpha : (-1, 1) \to \mathbb{R}^3$ be the space curve given by $\alpha(s) = \left(\frac{1}{3}(1+s)^{\frac{3}{2}}, \frac{1}{3}(1-s)^{\frac{3}{2}}, \frac{1}{\sqrt{2}}s\right)$, with unit tangent vector $\mathbf{T}(s) = \left(\frac{1}{2}(1+s)^{\frac{1}{2}}, -\frac{1}{2}(1-s)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}\right)$ and unit normal vector $\mathbf{N}(s) = \left(\sqrt{\frac{1-s}{2}}, \sqrt{\frac{1+s}{2}}, 0\right)$. Compute the unit binormal vector $\mathbf{B}(s)$ and the torsion τ.

2. Let $\mathbf{r}(s)$ be a regular arc length parametrized plane curve with curvature κ which is a constant.

   (a) Prove that $\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\mathbf{r}(s) + \dfrac{1}{\kappa}\mathbf{N}(s)\right) = \mathbf{0}$, where $\mathbf{N}$ is the unit normal vector.

   (b) Hence show that $\mathbf{r}(s)$ lies on a circle.

3. Let $\mathbf{r}(s)$ be a regular space curve with arc length parametrization, $\mathbf{T}(s)$ and $\mathbf{N}(s)$ be the unit tangent vector and unit normal vector respectively. Suppose $\kappa(s) > 0$ for any s and there exists a constant c and a constant unit vector $\mathbf{u}$ such that

$$\langle \mathbf{T}(s), \mathbf{u} \rangle = c$$

   for all s.

   (a) Show that $\mathbf{N}(s)$ and $\mathbf{u}$ are orthogonal for all s.

   (b) Using (a), show that there exists a constant θ such that $\mathbf{u} = \cos\theta\mathbf{T}(s) + \sin\theta\mathbf{B}(s)$ for all s.

   (c) Using (b) and the Frenet formulas, or otherwise, prove that $\frac{\tau(s)}{\kappa(s)} = \cot\theta$

2 Surfaces

1. The helicoid is parametrized by $\mathbf{x}(u, \theta) = (u\cos\theta, u\sin\theta, a\theta)$, for $a > 0, u, \theta \in \mathbb{R}$. Show that it is a regular surface.

2. Find the first fundamental form and the surface area of the following parametrized surface:

   (a) $\mathbf{x}(u, \theta) = (u\cos\theta, u\sin\theta, u^2)$, $u \in (0,1), \theta \in (0, 2\pi)$

   (b) $\mathbf{x}(u, \theta) = (u\cos\theta, u\sin\theta, \theta)$, $u \in (-1,1), \theta \in (0, 2\pi)$
   (You can use $\displaystyle\int \sqrt{x^2+1}\,dx = \frac{1}{2}(x\sqrt{x^2+1} + \ln(x + \sqrt{x^2+1})) + C$ directly)

3 Some tips/advice for Test 2

1. Test 2 Coverage: All lecture and tutorial materials up to the teaching content today, with emphasis on content after test 1

2. Revise things taught in the lecture and tutorial notes, including the examples (revising those not taught in class also helps, though not as important as those taught)

3. Do as much exercise from lecture notes (chapter 2) as possible to familiarise yourself with concepts such as curvature and Frenet frame

4. Understand how to prove a surface is regular and compute the first fundamental form and surface area